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Bosonization of vertex operators for the $A^{(1)}_{n-1}$ face model

Yoshinori Asai†, Michio Jimbo†, Tetsuji Miwa‡ and Yaroslav Pugai§

† Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606, Japan
‡ Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan
§ L D Landau Institute for Theoretical Physics, Chernogolovka, 142432, Russia

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Abstract. We present a free boson realization of the vertex operators and their duals for the solvable SOS lattice model of $A^{(1)}_{n-1}$ type. We discuss a possible connection with the calculation of correlation functions.

1. Introduction

The vertex operator approach [1–3] provides a powerful method for the study of correlation functions of solvable lattice models. It was originally formulated for vertex-type models, and then extended [4, 5] to incorporate face-type models such as the Andrews–Baxter–Forrester (ABF) model [6]. In particular it was shown that, in much the same way as with vertex models, correlation functions of the face models are given as traces of products of vertex operators, and are described in terms of a system of difference equations having the Boltzmann weights as coefficients. The most effective way of solving these difference equations is to realize the vertex operators in terms of bosonic free fields. For face-type models, such a realization had not been known, and it had remained an open question to give solutions to the difference equations. This problem was solved in a recent paper [7], on the basis of the ideas developed in [8, 9]. In particular, integral formulae were given for multi-point correlation functions of the ABF models.

The present paper can be viewed as a continuation of [7]. Here we deal with the $A^{(1)}_{n-1}$ face model [10], the ABF model being the $n = 2$ case. With the aid of the oscillators and screening currents introduced in [11, 12] in connection with the q-deformed W-algebras, we write down a bosonic realization of the vertex operators for the $A^{(1)}_{n-1}$ face model. In order to write down the correlation functions, we need the 'dual' vertex operators as well. This problem was absent for the ABF models, since the vertex operators are self-dual in that case. We construct such dual operators by the (skew-symmetric) fusion of the ordinary vertex operators. The construction of vertex operators and their duals is the main result of this paper. These operators are realized on a direct sum of Fock spaces, which is bigger than the actual space of states of the model. The latter should be realized as the cohomology of the BRST complex, as was done in the ABF case in [7]. We do not address this issue in this paper.

The text is organized as follows. In section 2 we recall the $A^{(1)}_{n-1}$ face model, and introduce the vertex operators along with their commutation relations. In section 3 we present the bosonization of the vertex operators and their duals. Section 4 is devoted to discussions and open problems. We defer some technical points to the appendices. In

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appendix A we give a graphical interpretation of the vertex operators and their duals, and explain how the correlation functions can be expressed in their terms. In appendix B we prove that the bosonic formulae for the vertex operators satisfy the correct commutation relations given in section 2. In appendix C we give a proof of the bosonization formula in section 3 for the dual vertex operators.

2. Commutation relations of vertex operators for the $A^{(1)}_{n-1}$ face model

Here we write the commutation relation of the vertex operators for the $A^{(1)}_{n-1}$ face model. We also construct the anti-symmetric fusion of vertex operators and derive their commutation relation.

2.1. The $A^{(1)}_{n-1}$ face model

After introducing notation, we recall the $A^{(1)}_{n-1}$ face model [10].

Let $\epsilon_\mu (1 \leqslant \mu \leqslant n)$ be the orthonormal basis in $\mathbb{R}^n$. We have the inner product $\langle \epsilon_\mu, \epsilon_\nu \rangle = \delta_{\mu\nu}$. Set

$$\bar{\epsilon}_\mu = \epsilon_\mu - \epsilon \qquad \epsilon = \frac{1}{n}\sum_{\mu=1}^{n} \epsilon_\mu \, . \tag{2.1}$$

The type $A^{(1)}_{n-1}$ weight lattice is the linear span of the $\bar{\epsilon}_\mu$

$$P = \sum_{\mu=1}^{n} \mathbb{Z}\bar{\epsilon}_\mu \, . \tag{2.2}$$

Note that $\sum_{\mu=1}^{n} \bar{\epsilon}_\nu = 0$. Let $\omega_\mu (1 \leqslant \mu \leqslant n-1)$ be the fundamental weights

$$\omega_\mu = \sum_{\nu=1}^{\mu} \bar{\epsilon}_\nu$$

and $\alpha_\mu (1 \leqslant \mu \leqslant n-1)$ the simple roots

$$\alpha_\mu = \epsilon_\mu - \epsilon_{\mu+1} \, .$$

For $a \in P$ we set

$$a_{\mu\nu} = \langle a + \rho, \epsilon_\mu - \epsilon_\nu \rangle$$

where $\rho = \sum_{\mu=1}^{n-1} \omega_\mu$.

An ordered pair $(b, a) \in P^2$ is called admissible if and only if there exists $\mu (1 \leqslant \mu \leqslant n)$ such that

$$b - a = \bar{\epsilon}_\mu \, .$$

We represent it as

$$b \xleftarrow{\ \mu\ } a \, .$$

An ordered set of four weights $(a, b, c, d) \in P^4$ is called an admissible configuration around a face if and only if the pairs (b, a), (c, b), (d, a) and (c, d) are admissible. We represent this as

$$\begin{tikzcd} c & d \arrow[l, "\kappa"'] \\ b \arrow[u, "\mu"] & a \arrow[l, "\nu"] \arrow[u, "\lambda"'] \end{tikzcd} \, .$$

Suppose that

$$b-a=\bar{\epsilon}_\nu \qquad c-b=\bar{\epsilon}_\mu \qquad d-a=\bar{\epsilon}_\lambda \qquad c-d=\bar{\epsilon}_\kappa\,. \tag{2.3}$$

There are three cases:

case (i) $\mu=\nu=\kappa=\lambda$

case (ii) $\mu=\lambda, \nu=\kappa \qquad (\mu\neq\nu)$

case (iii) $\mu=\kappa, \nu=\lambda \qquad (\mu\neq\nu)\,.$

With each admissible configuration around a face we associate the Boltzmann weight as follows.

We will use the following abbreviation:

$$[v]=x^{(v^2/r)-v}\Theta_{x^{2r}}(x^{2v})\,. \tag{2.4}$$

Here, r is an integer such that $r\geqslant n+2$, and x is a parameter such that $0<x<1$. We fix r, x throughout the paper. We have also used

$$\Theta_q(z)=(z;q)_\infty(qz^{-1};q)_\infty(q;q)_\infty \tag{2.5}$$

$$(z;q_1,\ldots,q_m)_\infty=\prod_{i_1,\ldots,i_m=0}^{\infty}(1-q_1^{i_1}\cdots q_m^{i_m}z)\,. \tag{2.6}$$

The Boltzmann weight associated with the configuration (2.3) is denoted by

$$W\left(\begin{matrix} c & d \\ b & a\end{matrix}\middle| v\right) \tag{2.7}$$

and is given by

$$W\left(\begin{matrix} a+2\bar{\epsilon}_\mu & a+\bar{\epsilon}_\mu \\ a+\bar{\epsilon}_\mu & a\end{matrix}\middle| v\right)=r_1(v) \tag{2.8}$$

$$W\left(\begin{matrix} a+\bar{\epsilon}_\mu+\bar{\epsilon}_\nu & a+\bar{\epsilon}_\mu \\ a+\bar{\epsilon}_\nu & a\end{matrix}\middle| v\right)=r_1(v)\frac{[v][a_{\mu\nu}-1]}{[v-1][a_{\mu\nu}]} \tag{2.9}$$

$$W\left(\begin{matrix} a+\bar{\epsilon}_\mu+\bar{\epsilon}_\nu & a+\bar{\epsilon}_\nu \\ a+\bar{\epsilon}_\nu & a\end{matrix}\middle| v\right)=r_1(v)\frac{[v-a_{\mu\nu}][1]}{[v-1][a_{\mu\nu}]}. \tag{2.10}$$

These weights satisfy the Yang–Baxter equation

$$\sum_g W\left(\begin{matrix} d & e \\ c & g\end{matrix}\middle| u_1\right)W\left(\begin{matrix} c & g \\ b & a\end{matrix}\middle| u_2\right)W\left(\begin{matrix} e & f \\ g & a\end{matrix}\middle| u_1-u_2\right) \tag{2.11}$$

$$=\sum_g W\left(\begin{matrix} g & f \\ b & a\end{matrix}\middle| u_1\right)W\left(\begin{matrix} d & e \\ g & f\end{matrix}\middle| u_2\right)W\left(\begin{matrix} d & g \\ c & b\end{matrix}\middle| u_1-u_2\right)\,. \tag{2.12}$$

The normalization factor $r_1(v)$ is determined by the condition that the partition function per face is equal to 1. The method of computation is standard (see, e.g., [13]). It is based on the following two equations called the inversion relations, which restrict $r_1(v)$.

The first inversion relation is

$$\sum_g W\left(\begin{matrix} c & g \\ b & a\end{matrix}\middle| -v\right)W\left(\begin{matrix} c & d \\ g & a\end{matrix}\middle| v\right)=\delta_{bd}\,. \tag{2.13}$$

The second inversion relation is

$$\sum_g G_g W\left(\begin{matrix} g & b \\ d & c \end{matrix}\middle| n-v\right) W\left(\begin{matrix} g & d \\ b & a \end{matrix}\middle| v\right) = \delta_{ac}\frac{G_b G_d}{G_a}\,. \tag{2.14}$$

where

$$G_a = \prod_{\mu<\nu}[a_{\mu\nu}]\,.$$

We consider the model in the so-called regime III, i.e. $0 < v < 1$. In this regime the partition function is given by the $m = 1$ case of the following definition:

$$r_m(v) = z^{((r-1)/r)(n-m)/n}\,\frac{g_m(z^{-1})}{g_m(z)} \qquad (z = x^{2v}) \tag{2.15}$$

$$g_m(z) = \frac{\{x^{m+1}z\}\{x^{2r+2n-m-1}z\}}{\{x^{2r+m-1}z\}\{x^{2n-m+1}z\}} \tag{2.16}$$

$$\{z\} = (z; x^{2r}, x^{2n})_\infty\,. \tag{2.17}$$

2.2. Commutation relations

Following the general principle of the algebraic approach in solvable lattice models, we give the commutation relation of the vertex operators for the $A^{(1)}_{n-1}$ face model.

Consider the operator symbol $\phi^{(b,a)}_\mu$ where $b = a + \mu$, and call it the vertex operator from a to b. In section 3, we will give a realization of $\phi^{(b,a)}_\mu$ in terms of bosons. In this section we treat them symbolically.

We consider the following commutation relation:

$$\phi^{(c,b)}_\mu(v_1)\phi^{(b,a)}_\nu(v_2) = \sum_d W\left(\begin{matrix} c & d \\ b & a \end{matrix}\middle| v_1 - v_2\right)\phi^{(c,d)}_\kappa(v_2)\phi^{(d,a)}_\lambda(v_1)\,. \tag{2.18}$$

Note that for case (i) for (2.3) the sum is only for $d = b$, while for cases (ii) and (iii) they mix together.

In appendix A we give the identification of (2.18) with the commutation relation of the half transfer matrix, which motivates our investigation. The aim of this paper is to give a bosonization of (2.18). This is done in section 3.

2.3. Fusion of the Boltzmann weights

We define the fused Boltzmann weights and give the relation between the fused weights and the original ones.

Let us define three types of fused weights. The first one is obtained by fusion in the horizontal direction, the second in the vertical direction, and the third in both directions. We first prepare three types of admissible configurations around a face.

Definition 2.1.

$$(a, b, c, d) \in P^4 \text{ is h-admissible} \tag{2.19}$$

$$\stackrel{\text{def}}{\Longleftrightarrow} b - a = -\bar{\epsilon}_\nu,\, c - b = \bar{\epsilon}_\mu,\, d - a = \bar{\epsilon}_\kappa,\, c - d = -\bar{\epsilon}_\lambda \qquad \text{for some } \nu, \mu, \kappa, \lambda.$$

$$(a, b, c, d) \in P^4 \text{ is v-admissible} \tag{2.20}$$

$$\overset{\text{def}}{\Longleftrightarrow} b-a=\bar{\epsilon}_\nu,\, c-b=-\bar{\epsilon}_\mu,\, d-a=-\bar{\epsilon}_\kappa,\, c-d=\bar{\epsilon}_\lambda \qquad \text{for some } \nu,\mu,\kappa,\lambda.$$

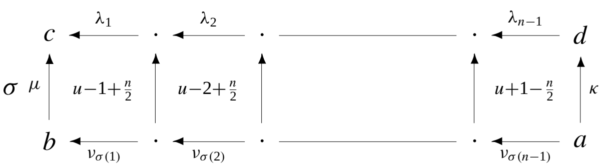
$$(a,b,c,d)\in P^4 \text{ is } *\text{-admissible} \tag{2.21}$$

$$\overset{\text{def}}{\Longleftrightarrow} b-a=-\bar{\epsilon}_\nu,\, c-b=-\bar{\epsilon}_\mu,\, d-a=-\bar{\epsilon}_\kappa,\, c-d=-\bar{\epsilon}_\lambda \qquad \text{for some } \nu,\mu,\kappa,\lambda.$$

Given $\nu\in\{1,\dots,n\}$, let $\nu_1,\dots,\nu_{n-1}\in\{1,\dots,n\}$ be such that $\nu_1<\cdots<\nu_{n-1}$ and $\{\nu,\nu_1,\dots,\nu_{n-1}\}=\{1,\dots,n\}$. We denote $(\nu_1,\dots,\nu_{n-1})$ by $\hat{\nu}$. Note that $b-a=-\bar{\epsilon}_\nu$ is equivalent to $b-a=\sum_{i=1}^{n-1}\bar{\epsilon}_{\nu_i}$. We represent it graphically as

$$b \xleftarrow{\;\hat{\nu}=(\nu_1,\dots,\nu_{n-1})\;} a\,.$$

For a h-admissible quadruple (a,b,c,d), define the fused Boltzmann weight W_{h} as follows:

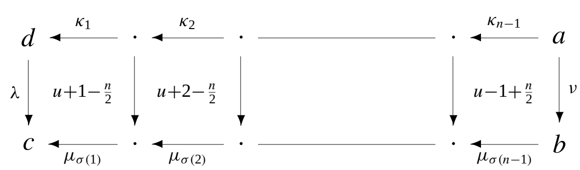
$$W_{\mathrm{h}}\left(\begin{matrix} c & d\\ b & a\end{matrix}\middle| u\right)=\sum_{\sigma\in\mathcal{S}_{n-1}}\operatorname{sgn}\sigma$$


$$=w\sum_{\sigma\in\mathcal{S}_{n-1}}\operatorname{sgn}\sigma\prod_{i=1}^{n-1}W\left(\begin{matrix} c_i & c_{i+1}\\ b_i^\sigma & b_{i+1}^\sigma\end{matrix}\middle| u+\frac{n}{2}-i\right) \tag{2.22}$$

where

$$\begin{aligned} c_i&=c-\bar{\epsilon}_{\lambda_1}-\cdots-\bar{\epsilon}_{\lambda_{i-1}} \qquad (c_1=c,\, c_n=d)\\ b_i^\sigma&=b-\bar{\epsilon}_{\nu_{\sigma(1)}}-\cdots-\bar{\epsilon}_{\nu_{\sigma(i-1)}} \qquad (b_1^\sigma=b,\, b_n^\sigma=a)\,. \end{aligned} \tag{2.23}$$

Likewise, for a v-admissible (a,b,c,d), define

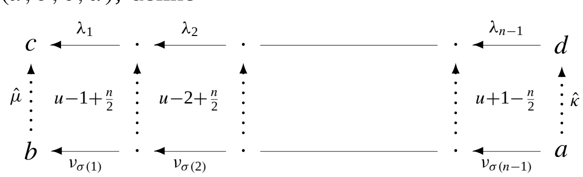
$$W_{\mathrm{v}}\left(\begin{matrix} c & d\\ b & a\end{matrix}\middle| u\right)=\sum_{\sigma\in\mathcal{S}_{n-1}}\operatorname{sgn}\sigma$$


$$=\sum_{\sigma\in\mathcal{S}_{n-1}}\operatorname{sgn}\sigma\prod_{j=1}^{n-1}W\left(\begin{matrix} c_j^\sigma & d_j\\ c_{j+1}^\sigma & d_{j+1}\end{matrix}\middle| u-\frac{n}{2}+j\right) \tag{2.24}$$

where

$$\begin{aligned} c_j^\sigma&=c-\bar{\epsilon}_{\mu_{\sigma(1)}}-\cdots-\bar{\epsilon}_{\mu_{\sigma(j-1)}}\\ d_j&=d-\bar{\epsilon}_{\kappa_1}-\cdots-\bar{\epsilon}_{\kappa_{j-1}}. \end{aligned} \tag{2.25}$$

Finally, for a $*$-admissible (a,b,c,d), define

$$W_*(\begin{matrix} c & d\\ b & a\end{matrix}|u)=\sum_{\sigma\in\mathcal{S}_{n-1}}\operatorname{sgn}\sigma$$


$$=\sum_{\sigma\in\mathcal{S}_{n-1}}\operatorname{sgn}\sigma\prod_{i=1}^{n-1}W_{\mathrm{v}}\left(\begin{matrix} c_i & c_{i+1}\\ b_i^\sigma & b_{i+1}^\sigma\end{matrix}\middle| u+\frac{n}{2}-i\right) \tag{2.26}$$

$$
= \sum_{\sigma\in\mathcal{S}_{n-1}} \operatorname{sgn}\sigma
\begin{tikzcd}
d \arrow[d, dotted, "\hat\lambda"'] \arrow[dr, phantom, "u+1-\frac{n}{2}"] & \cdot \arrow[l, "\kappa_1"'] \arrow[d, dotted] \arrow[dr, phantom, "u+2-\frac{n}{2}"] & \cdot \arrow[l, "\kappa_2"'] \arrow[d, dotted] & \cdot \arrow[l, dash] \arrow[d, dotted] \arrow[dr, phantom, "u-1+\frac{n}{2}"] & a \arrow[l, "\kappa_{n-1}"'] \arrow[d, dotted, "\hat\nu"] \\
c & \cdot \arrow[l, "\mu_{\sigma(1)}"] & \cdot \arrow[l, "\mu_{\sigma(2)}"] & \cdot \arrow[l, dash] & b \arrow[l, "\mu_{\sigma(n-1)}"]
\end{tikzcd}
$$

$$
= \sum_{\sigma\in\mathcal{S}_{n-1}} \operatorname{sgn}\sigma \prod_{j=1}^{n-1} W_{\mathrm{h}} \left(\begin{matrix} c_j^\sigma & d_j \\ c_{j+1}^\sigma & d_{j+1} \end{matrix} \middle| u - \frac{n}{2} + j \right) . \tag{2.27}
$$

Proposition 2.2. We have the following formulae for the fused Boltzmann weights.:

$$
\begin{tikzcd}
c \arrow[dr, phantom, "\frac{n}{2}-u"] & d \arrow[l, dotted, "\hat\lambda"'] \\
b \arrow[u, "\mu"] & a \arrow[l, dotted, "\hat\nu"] \arrow[u, "\kappa"']
\end{tikzcd}
= (-1)^{\lambda+\nu+n-1} \frac{G_b}{G_c}
\begin{tikzcd}
c \arrow[r, "\lambda"] \arrow[dr, phantom, "u"] & d \\
b \arrow[u, "\mu"] \arrow[r, "\nu"'] & a \arrow[u, "\kappa"']
\end{tikzcd}
\tag{2.28}
$$

$$
\begin{tikzcd}
c \arrow[dr, phantom, "\frac{n}{2}-u"] & d \arrow[l, "\lambda"'] \\
b \arrow[u, dotted, "\hat\mu"] & a \arrow[l, "\nu"] \arrow[u, dotted, "\hat\kappa"']
\end{tikzcd}
= (-1)^{\mu+\kappa+n-1} \frac{G_b}{G_a}
\begin{tikzcd}
c \arrow[d, "\mu"'] \arrow[dr, phantom, "u"] & d \arrow[l, "\lambda"'] \arrow[d, "\kappa"] \\
b & a \arrow[l, "\nu"]
\end{tikzcd}
\tag{2.29}
$$

$$
\begin{tikzcd}
c \arrow[dr, phantom, "u"] & d \arrow[l, dotted, "\hat\lambda"'] \\
b \arrow[u, dotted, "\hat\mu"] & a \arrow[l, dotted, "\hat\nu"] \arrow[u, dotted, "\hat\kappa"']
\end{tikzcd}
= \frac{G_b}{G_d}
\begin{tikzcd}
c \arrow[r, "\lambda"] \arrow[d, "\mu"'] \arrow[dr, phantom, "u"] & d \arrow[d, "\kappa"] \\
b \arrow[r, "\nu"'] & a
\end{tikzcd}
. \tag{2.30}
$$

For instance, equation (2.28) means that

$$
W_{\mathrm{h}} \left(\begin{matrix} c & d \\ b & a \end{matrix} \middle| \frac{n}{2} - u \right) = (-1)^{\lambda+\nu+n-1} \frac{G_b}{G_c} W \left(\begin{matrix} d & a \\ c & b \end{matrix} \middle| u \right) .
$$

In the calculation we used

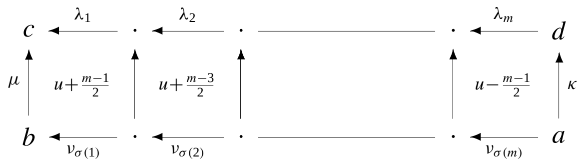
$$
\prod_{i=1}^{n-1} r_1 \left(u + i - \frac{n}{2} \right) = (-1)^{n-1} \frac{[\frac{n-2}{2} - u]}{[\frac{n-2}{2} + u]} r_{n-1}(u) \tag{2.31}
$$

$$
\prod_{i=1}^{n-1} r_{n-1} \left(u + i - \frac{n}{2} \right) = r_1(u) \tag{2.32}
$$

$$
r_{n-1}(u) \frac{[\frac{n-2}{2} - u]}{[u - \frac{n}{2}]} = r_1 \left(\frac{n}{2} - u \right) . \tag{2.33}
$$

In the above, we considered the anti-symmetric fusion of $n-1$ times. In appendix C we also need the anti-symmetric fusion of m times for $2 \leqslant m \leqslant n-2$. The definition of weight reads as

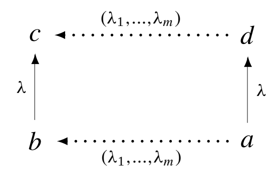
$$
\begin{tikzcd}[column sep=large]
c & d \arrow[l, dotted, "(\lambda_1,\ldots,\lambda_m)"'] \\
b \arrow[u, "\mu"] & a \arrow[l, dotted, "(\nu_1,\ldots,\nu_m)"] \arrow[u, "\kappa"']
\end{tikzcd}
$$

$$= \sum_{\sigma \in S_m} \operatorname{sgn} \sigma$$  $$. \qquad (2.34)$$

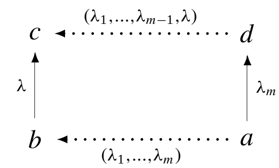
It is anti-symmetric with respect to $(\nu_1, \ldots, \nu_m)$ by the definition. In fact, it is also anti-symmetric with respect to $(\lambda_1, \ldots, \lambda_m)$.

The vertical fusion is similarly defined by taking $u - \frac{m-1}{2}, u - \frac{m-3}{2}, \ldots, u + \frac{m-1}{2}$ to be the spectral parameters (see (2.24) for the case $m = n - 1$).

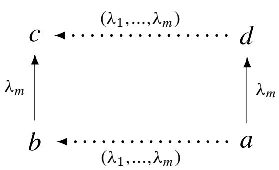
The result is as follows:



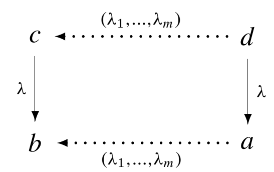
$$= (-1)^{m-1} r_m(u) \frac{[u - \frac{m-1}{2}]}{[u - \frac{m+1}{2}]} \prod_{i=1}^{m} \frac{[a_{\lambda\lambda_i} - 1]}{[a_{\lambda\lambda_i}]} \qquad (2.35)$$



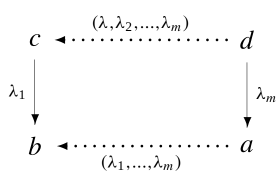
$$= (-1)^{m-1} r_m(u) \frac{[u - \frac{m-1}{2} - a_{\lambda\lambda_m}][1]}{[u - \frac{m+1}{2}][a_{\lambda\lambda_m}]} \prod_{i=1}^{m-1} \frac{[a_{\lambda\lambda_i} - 1]}{[a_{\lambda\lambda_i}]} \qquad (2.36)$$



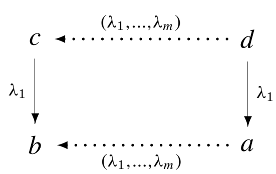
$$= (-1)^{m-1} r_m(u) \prod_{i=1}^{m-1} \frac{[a_{\lambda_m\lambda_i}]}{[a_{\lambda_m\lambda_i} + 1]} \qquad (2.37)$$



$$= (-1)^{m-1} r_m(u) \frac{[u - \frac{m-1}{2}]}{[u - \frac{m+1}{2}]} \prod_{i=1}^{m} \frac{[a_{\lambda\lambda_i} + 1]}{[a_{\lambda\lambda_i}]} \qquad (2.38)$$



$$= (-1)^{m} r_m(u) \frac{[u - \frac{m-1}{2} + a_{\lambda\lambda_1}][1]}{[u - \frac{m+1}{2}][a_{\lambda\lambda_1}]} \prod_{i=2}^{m} \frac{[a_{\lambda\lambda_i} + 1]}{[a_{\lambda\lambda_i}]} \qquad (2.39)$$



$$= (-1)^{m-1} r_m(u) \prod_{i=2}^{m} \frac{[a_{\lambda_1\lambda_i} + 1]}{[a_{\lambda_1\lambda_i}]} \, . \qquad (2.40)$$

2.4. Dual vertex operators and commutation relations

We introduce the dual vertex operators $\phi^*_\lambda(v)$ and write down the commutation relations between them in terms of the fused Boltzmann weights.

Definition 2.3. Suppose that $(\lambda_1, \ldots, \lambda_m)$ are distinct and $b - a = \sum_{i=1}^m \bar{\epsilon}_{\lambda_i}$. We define

$$\phi^{(b,a)}_{(\lambda_1,\ldots,\lambda_m)}(v) = \sum_{\sigma\in\mathcal{S}_m} \operatorname{sgn}\sigma\, \phi_{\lambda_{\sigma(1)}}\left(v - \frac{m-1}{2}\right)\phi_{\lambda_{\sigma(2)}}\left(v - \frac{m-3}{2}\right)$$

$$\cdots \phi_{\lambda_{\sigma(m)}}\left(v + \frac{m-1}{2}\right)$$

$$= \begin{vmatrix} \phi_{\lambda_1}\left(v - \frac{m-1}{2}\right) & \phi_{\lambda_1}\left(v - \frac{m-3}{2}\right) & \cdots & \phi_{\lambda_1}(v + \frac{m-1}{2}) \\ \phi_{\lambda_2}\left(v - \frac{m-1}{2}\right) & \phi_{\lambda_2}\left(v - \frac{m-3}{2}\right) & \cdots & \phi_{\lambda_2}(v + \frac{m-1}{2}) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{\lambda_m}\left(v - \frac{m-1}{2}\right) & \phi_{\lambda_m}\left(v - \frac{m-3}{2}\right) & \cdots & \phi_{\lambda_m}(v + \frac{m-1}{2}) \end{vmatrix}. \tag{2.41}$$

We also define

$$\phi^{*(b,a)}_{\lambda}(v) = \phi^{(b,a)}_{\hat{\lambda}}(v)\,. \tag{2.42}$$

Proposition 2.4. The notation being as in (2.19)–(2.21), we have

$$\phi^{(c,b)}_{\mu}(v_1)\phi^{*(b,a)}_{\nu}(v_2) = \sum_d W_{\mathrm{h}}\left(\begin{matrix} c & d \\ b & a \end{matrix}\middle| v_1 - v_2\right)\phi^{*(c,d)}_{\lambda}(v_2)\phi^{(d,a)}_{\kappa}(v_1) \tag{2.43}$$

$$\phi^{*(c,b)}_{\mu}(v_1)\phi^{(b,a)}_{\nu}(v_2) = \sum_d W_{\mathrm{v}}\left(\begin{matrix} c & d \\ b & a \end{matrix}\middle| v_1 - v_2\right)\phi^{(c,d)}_{\lambda}(v_2)\phi^{*(d,a)}_{\kappa}(v_1) \tag{2.44}$$

$$\phi^{*(c,b)}_{\mu}(v_1)\phi^{*(b,a)}_{\nu}(v_2) = \sum_d W_{*}\left(\begin{matrix} c & d \\ b & a \end{matrix}\middle| v_1 - v_2\right)\phi^{*(c,d)}_{\lambda}(v_2)\phi^{*(d,a)}_{\kappa}(v_1)\,. \tag{2.45}$$

3. Bosonization of vertex operators

3.1. Bosons

Consider the bosonic oscillators β^j_m $(1 \leqslant j \leqslant n-1, m \in \mathbb{Z}\backslash\{0\})$ with the commutation relations

$$[\beta^j_m, \beta^k_{m'}] = m\frac{[(n-1)m]_x}{[nm]_x}\frac{[(r-1)m]_x}{[rm]_x}\delta_{m+m',0} \qquad (j = k) \tag{3.1}$$

$$= -mx^{sgn(j-k)nm}\frac{[m]_x}{[nm]_x}\frac{[(r-1)m]_x}{[rm]_x}\delta_{m+m',0} \qquad (j \neq k)\,. \tag{3.2}$$

Here the symbol $[a]_x$ stands for $(x^a - x^{-a})/(x - x^{-1})$. Define β^n_m by

$$\sum_{j=1}^n x^{-2jm}\beta^j_m = 0\,. \tag{3.3}$$

Then the commutation relations (3.1), (3.2) are valid for all $1 \leqslant j, k \leqslant n$. These oscillators were introduced in [11, 12].

We also introduce the zero mode operators P_α, Q_α indexed by $\alpha \in P = \oplus_{i=1}^{n-1}\mathbb{Z}\omega_i$. By definition they are $\mathbb{Z}$-linear in α and satisfy

$$[iP_\alpha, Q_\beta] = \langle\alpha, \beta\rangle \qquad (\alpha, \beta \in P)\,.$$

We shall deal with the bosonic Fock spaces $\mathcal{F}_{l,k}$ $(l, k \in P)$ generated by β^j_{-m} $(m > 0)$ over the vacuum vectors $|l, k\rangle$:

$$\mathcal{F}_{l,k} = \mathbb{C}[\{\beta^j_{-1}, \beta^j_{-2}, \ldots\}_{1\leqslant j\leqslant n}]|l, k\rangle$$

where

$$\beta^j_m|k, l\rangle = 0 \qquad (m > 0)$$

$$P_\alpha|l, k\rangle = \left\langle \alpha, \sqrt{\frac{r}{r-1}}l - \sqrt{\frac{r-1}{r}}k \right\rangle |l, k\rangle$$

$$|l, k\rangle = \exp\left(\mathrm{i}\sqrt{\frac{r}{r-1}}Q_l - \mathrm{i}\sqrt{\frac{r-1}{r}}Q_k\right)|0, 0\rangle .$$

3.2. Basic operators

For $j = 1, \ldots, n-1$ define

$$U_{-\alpha_j}(z) = \exp\left(\mathrm{i}\sqrt{\frac{r-1}{r}}(Q_{\alpha_j} - \mathrm{i}P_{\alpha_j}\log z)\right) : \exp\left(\sum_{m\neq 0}\frac{1}{m}(\beta^j_m - \beta^{j+1}_m)(x^j z)^{-m}\right) : \tag{3.4}$$

$$U_{\omega_j}(z) = \exp\left(-\mathrm{i}\sqrt{\frac{r-1}{r}}(Q_{\omega_j} - \mathrm{i}P_{\omega_j}\log z)\right) : \exp\left(-\sum_{m\neq 0}\frac{1}{m}\sum_{k=1}^{j} x^{(j-2k+1)m}\beta^k_m z^{-m}\right) : . \tag{3.5}$$

Note that

$$\exp\left(\mathrm{i}\sqrt{\frac{r-1}{r}}(Q_\beta - \mathrm{i}P_\beta\log z)\right) = z^{((r-1)/2r)\langle\beta,\beta\rangle} \exp\left(\mathrm{i}\sqrt{\frac{r-1}{r}}Q_\beta\right)z^{\sqrt{(r-1)/r}\,P_\beta} .$$

Up to a power of z, the operators $U_{-\alpha_j}(z)$ are the screening currents for the quantum W-algebras in the sense of [11, 12]. In view of (3.3) and $\omega_n = \bar\epsilon_1 + \cdots + \bar\epsilon_n = 0$, we set $U_{\omega_n}(z) = 1$. We shall often use the variable v such that $z = x^{2v}$, and write

$$\xi_j(v) = U_{-\alpha_j}(z) \qquad \eta_j(v) = U_{\omega_j}(z) .$$

We shall need the following commutation relations between them:

$$\eta_1(v)\eta_j(v') = r_j(v - v')\eta_j(v')\eta_1(v) \tag{3.6}$$

$$\xi_j(v)\eta_j(v') = -f(v - v', 0)\eta_j(v')\xi_j(v) \tag{3.7}$$

$$\xi_j(v)\xi_{j+1}(v') = -f(v - v', 0)\xi_{j+1}(v')\xi_j(v) \tag{3.8}$$

$$\xi_j(v)\xi_j(v') = h(v - v')\xi_j(v')\xi_j(v) . \tag{3.9}$$

All other combinations mutually commute, except for $\eta_j(v)\eta_k(v')$. Here $r_j(v)$ is given by (2.15), and

$$f(v, w) = \frac{[v + \frac{1}{2} - w]}{[v - \frac{1}{2}]} \tag{3.10}$$

$$h(v) = \frac{[v - 1]}{[v + 1]} . \tag{3.11}$$

3.3. Vertex operators

In what follows we set

$$\pi_\mu = \sqrt{r(r-1)}P_{\bar{\epsilon}_\mu} \qquad \pi_{\mu\nu} = \pi_\mu - \pi_\nu\,.$$

Then $\pi_{\mu\nu}$ acts on $\mathcal{F}_{l,k}$ as an integer $\langle \epsilon_\mu - \epsilon_\nu, rl-(r-1)k\rangle$.

The commutation relations presented in section 2 can be realized in the form of screened vertex operators. For $\mu = 1, \ldots, n$ define

$$\phi_\mu(v) = \oint \prod_{j=1}^{\mu-1} \frac{\mathrm{d}z_j}{2\pi \mathrm{i} z_j} \eta_1(v)\xi_1(v_1)\cdots\xi_{\mu-1}(v_{\mu-1}) \prod_{j=1}^{\mu-1} f(v_j - v_{j-1}, \pi_{j\mu}) \tag{3.12}$$

$$= (-1)^{\mu-1} \oint \prod_{j=1}^{\mu-1} \frac{\mathrm{d}z_j}{2\pi \mathrm{i} z_j} \xi_{\mu-1}(v_{\mu-1})\cdots\xi_1(v_1)\eta_1(v) \prod_{j=1}^{\mu-1} f(v_{j-1} - v_j, 1-\pi_{j\mu})\,. \tag{3.13}$$

Here we set $v_0 = v$, $z_j = x^{2v_j}$. The equality of (3.12) and (3.13) follows from the commutation relations (3.6)–(3.9) and

$$f(v,0)f(-v,w) = f(v,1-w)\,.$$

From the contraction rules of $\xi_j(v), \eta_j(v)$ given in (C.1)–(C.7), we find that on each $\mathcal{F}_{l,k}$ the integrand of (3.12) comprises only integral powers of z_j $(1 \leqslant j \leqslant \mu-1)$, and that it has poles at $z_j = x^{1+2rk}z_{j-1}, x^{-1-2rk}z_{j-1}$ $(k = 0, 1, 2, \ldots)$. We take the integration contours to be simple closed curves around the origin satisfying

$$x|z_{j-1}| < |z_j| < x^{-1}|z_{j-1}| \qquad (j = 1, \ldots, \mu-1)\,.$$

Theorem 3.1. The operators (3.12) satisfy the commutation relations (2.18).

The proof is given in appendix B.

Likewise define

$$\bar{\phi}^{*(m-1)}_\mu(v) = c_m^{-1} \oint \prod_{j=\mu}^{m-1} \frac{\mathrm{d}z_j}{2\pi \mathrm{i} z_j} \eta_{m-1}(v)\xi_{m-1}(v_{m-1})\cdots\xi_\mu(v_\mu) \prod_{j=\mu+1}^{m} f(v_{j-1} - v_j, \pi_{\mu j}) \tag{3.14}$$

$$= (-1)^{m-\mu} c_m^{-1} \oint \prod_{j=\mu}^{m-1} \frac{\mathrm{d}z_j}{2\pi \mathrm{i} z_j} \xi_\mu(v_\mu) \cdots\xi_{m-1}(v_{m-1})\eta_{m-1}(v) \prod_{j=\mu+1}^{m} f(v_j - v_{j-1}, 1-\pi_{\mu j}) \tag{3.15}$$

where $v_m = v$ and

$$x|z_{j+1}| < |z_j| < x^{-1}|z_{j+1}| \qquad (j = \mu, \ldots, m-1)\,.$$

For convenience we have included a constant c_j given by

$$c_j = x^{((r-1)/r)j(j-1)/2n} \frac{g_{j-1}(x^j)}{(x^2;x^{2r})^j_\infty (x^{2r};x^{2r})^{2j-3}_\infty}\,. \tag{3.16}$$

Consider the 'fused' operators defined as minor determinants of $\phi_\mu(v)$ (see equation (2.41)):

$$\phi^{*(m-1)}_\mu(v) = \left(\prod_{j=1}^{m} c_j^{-1}\right) \phi_{(1,\ldots,\mu-1,\mu+1,\ldots,m)}(v) \qquad (1 \leqslant \mu \leqslant m)\,. \tag{3.17}$$

What follows gives an explicit formula for these quantities.

Theorem 3.2. For $2 \leqslant m \leqslant n$ we have

$$\phi_\mu^{*(m-1)}(v) = \bar{\phi}_\mu^{*(m-1)}(v) \prod_{\substack{1\leqslant\kappa<\lambda\leqslant m \\ \kappa,\lambda\neq\mu}} [\pi_{\kappa\lambda}] \,. \tag{3.18}$$

In addition, we have the following inversion identities.

Theorem 3.3.

$$\begin{vmatrix} \phi_1\left(v-\frac{n-1}{2}\right) & \phi_1\left(v-\frac{n-3}{2}\right) & \cdots & \phi_1(v+\frac{n-1}{2}) \\ \phi_2\left(v-\frac{n-1}{2}\right) & \phi_2\left(v-\frac{n-3}{2}\right) & \cdots & \phi_2(v+\frac{n-1}{2}) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_n\left(v-\frac{n-1}{2}\right) & \phi_n\left(v-\frac{n-3}{2}\right) & \cdots & \phi_n(v+\frac{n-1}{2}) \end{vmatrix} = \prod_{1\leqslant\kappa<\lambda\leqslant n} [\pi_{\kappa\lambda}] \times \mathrm{id} \,. \tag{3.19}$$

In terms of $\bar{\phi}_\mu^{*(n-1)}(v)$, equation (3.19) can alternatively be written in either of the following ways:

$$\sum_{\mu=1}^{n} \phi_\mu\left(v-\frac{n}{2}\right) \bar{\phi}_\mu^{*(n-1)}(v) \prod_{\substack{1\leqslant\lambda\leqslant n \\ \lambda\neq\mu}} [\pi_{\mu\lambda}]^{-1} = \mathrm{id} \tag{3.20}$$

$$\sum_{\mu=1}^{n} \bar{\phi}_\mu^{*(n-1)}\left(v-\frac{n}{2}\right) \phi_\mu(v) \prod_{\substack{1\leqslant\lambda\leqslant n \\ \lambda\neq\mu}} [\pi_{\lambda\mu}]^{-1} = \mathrm{id} \,. \tag{3.21}$$

The proofs of theorems 3.2 and 3.3 will be given in appendix C.

4. Discussion

In this paper we have constructed a free boson realization of the vertex operators $\phi_\mu(u)$, $\phi_\mu^*(u)$ for the $A_{n-1}^{(1)}$ face model. As is explained in appendix A, these operators correspond to the half-infinite transfer matrices on the lattice, and are designed to satisfy the same commutation relations as the latter. There is, however, a serious difference between the two which does not allow us to identify them directly. The operators $\phi_\mu(u)$, $\phi_\mu^*(u)$ are acting on the direct sum $\mathcal{F} = \oplus_{l,k\in P}\mathcal{F}_{l,k}$ of bosonic Fock spaces. On the other hand, the half-transfer matrices act on the eigenspaces $\mathcal{H}_{l,k}$ of the corner transfer matrices. The problem is that the character of $\mathcal{H}_{l,k}$ is different from that of $\mathcal{F}_{l,k}$. This is particularly significant for the calculation of correlation functions, since they are given as the trace of products of vertex operators over the 'true' space of states $\mathcal{H}_{l,k}$, rather than $\mathcal{F}_{l,k}$.

Let us discuss this point by taking the case $n = 2$. In the conformal limit $x = 1$, $\mathcal{H}_{l,k}$ becomes the irreducible minimal unitary modules over the Virasoro algebra, and the vertex operators become the chiral primary fields associated with them. In order to realize these representations, we need to introduce the BRST charge operator $Q : \mathcal{F} \to \mathcal{F}$, $Q^2 = 0$, and consider Felder's [14] complex

$$\cdots \xrightarrow{Q} \mathcal{F}_{l_{-1},k} \xrightarrow{Q} \mathcal{F}_{l_0,k} \xrightarrow{Q} \mathcal{F}_{l_1,k} \xrightarrow{Q} \cdots \,.$$

In this complex, only the 0th cohomology is non-trivial and gives the irreducible module $\mathcal{H}_{l,k}$. At the same time, the BRST charge Q commutes with the vertex operators so that the latter are well-defined as operators on $\mathcal{H}_{l,k}$. In the work [7], the construction of the BRST complex was carried over to the deformed case $0 < x < 1$ with $n = 2$ (see also [15]). Thanks to the commutativity of Q and the vertex operators, the calculation of the trace over

$\mathcal{H}_{l,k}$ is reduced to that over the Fock spaces by the Euler–Poincaré principle. In this way, it was possible in [7] to derive an explicit integral formula for the correlation functions. Thus the construction of the BRST complex and the calculation of the cohomology for $n \geqslant 3$ is the remaining important problem. This seems to be a rather non-trivial matter, and to our knowledge, has not been completely settled even in the conformal limit $x = 1$ (see, e.g., [16]).

In this connection, we note that in the conformal case $x = 1$ the spaces $\mathcal{H}_{l,k}$ are identified as irreducible representations of the W_n-algebra [17, 18]. A q-deformation of the W_n algebra was introduced in [11, 12], where it is shown that the screening currents employed in the present paper commute with the generators of the q-deformed W-algebras up to a total difference. Therefore, we naturally expect that the deformed W-algebra plays the role of the symmetry algebra for the lattice model, the vertex operators being the q-analogue of the chiral primary fields.

The vertex operators considered in this paper are 'type I operators' in the terminology of [1–3]. To complete the picture, it would also be interesting to study the bosonization of type II vertex operators which are responsible for creation/annihilation of excitations in the lattice model.

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Appendix A. Graphical definition of the vertex operators

In this appendix we outline the method of computing the correlation functions in the RSOS models with the Boltzmann weights (2.8), (2.9), (2.10). In the restricted model, the restriction on a is such that $a \in P^+_{r-n}$ where

$$P_{+l} = \left\{ a = \sum_{i=1}^{n-1} a_i \omega_i \in P; a_i \geqslant 0, \sum_{i=1}^{n-1} a_i \leqslant r - n \right\} .$$

We consider regime III, i.e. the case $0 < v < 1$. In this regime the ground-state configurations are parametrized by $b \in P^+_{r-n-1}$; a ground-state configuration is one which consists of b, $b + \omega_1$, $b + \omega_2, \ldots, b + \omega_{n-1}$. We choose and fix b. In our notation, we often drop the b-dependence.

The corner transfer matrices $A^{(a)}(v)$, $B^{(a)}(v)$, $C^{(a)}(v)$, $D^{(a)}(v)$ are associated with the four quadrants separated at the centre that takes a fixed state a.

$$\begin{array}{c|c} C^{(a)}(v) & B^{(a)}(v) \\ \hline \multicolumn{2}{c}{a} \\ \hline D^{(a)}(v) & A^{(a)}(v) \end{array}$$

The partition function Z is formally given by

$$Z = \sum_{a \in P^+_{r-n}} \operatorname{tr} D^{(a)}(v) C^{(a)}(v) B^{(a)}(v) A^{(a)}(v) .$$

In the large lattice limit, apart from a divergent scalar (independent of a), the CTM $A^{(a)}(v)$ is of the form

$$A^{(a)}(v) \sim x^{2vH}$$

where the operator H is independent of v. Let us denote by $\mathcal{H}_{l,k}$, where

$$l = b + \rho \qquad k = a + \rho$$

the space spanned by the eigenvectors of $A^{(a)}(v)$ with the boundary condition given by the choice of $b \in P^{+}_{r-n-1}$. By using equation (2.14), we also have

$$D^{(a)}(v)C^{(a)}(v)B^{(a)}(v)A^{(a)}(v) \sim G_a x^{2nH} .$$

The spectrum of H is obtained in [10]. Choosing the normalization of H appropriately we have

$$\mathrm{tr}_{\mathcal{H}_{l,k}} q^H = \chi_{l,k}(q)$$

where

$$\chi_{l,k}(q) = q^{(1-n)/24}(q;q)_\infty^{1-n} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\theta_{rl-(r-1)\sigma(k),r(r-1)}(q) \tag{A.1}$$

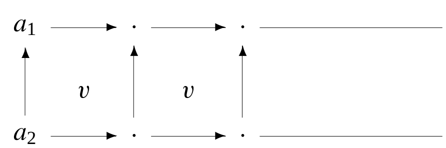
$$\theta_{\mu,m}(q) = \sum_{\alpha \in \sum_{j=1}^{n-1} \mathbb{Z}\alpha_j} q^{(m/2)|\alpha+\mu/m|^2} . \tag{A.2}$$

Note that

$$\chi_{l,k}(q) = q^{(|rl-(r-1)k|^2/2r(r-1))+(1-n)/24}\ (1 + \mathrm{O}(q)) . \tag{A.3}$$

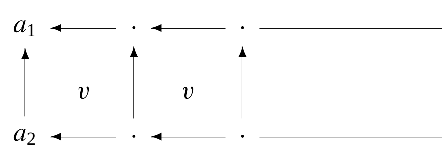
In order to compute the correlation functions, we use the half transfer matrices. There are four kinds of half transfer matrices that are extending to the north, east, west and south directions:

$\Phi_N^{(a_1,a_2)}(v)$:



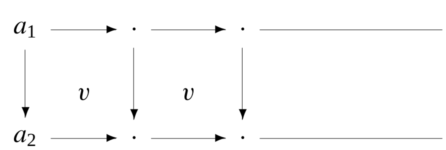
(A.4)

$\Phi_E^{(a_1,a_2)}(v)$:



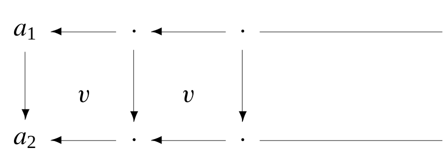
(A.5)

$\Phi_W^{(a_1,a_2)}(v)$:



(A.6)

$\Phi_S^{(a_1,a_2)}(v)$:



. (A.7)

Let $P(a_1,\ldots,a_m)$ be the probability that the local states of successive m sites, say from 1 to m, on the same column, take the values $a_1,\ldots,a_m$, respectively. We have

$$P(a_1,\ldots,a_m)=\frac{1}{Z}\,\mathrm{tr}_{\mathcal{H}_{l,a_m+\rho}}\,D^{(a_m)}(v)\Phi_W^{(a_m,a_{m-1})}(v)\cdots\Phi_W^{(a_2,a_1)}(v)$$
$$\times C^{(a_1)}(v)B^{(a_1)}(v)\Phi_E^{(a_1,a_2)}(v)\cdots\Phi_E^{(a_{m-1},a_m)}(v)A^{(a_m)}(v)\,. \qquad \text{(A.8)}$$

Using the equalities

$$\Phi_E^{(a,a')}(v)A^{(a')}(v)=A^{(a)}(v)\Phi_E^{(a,a')}(0) \qquad \text{(A.9)}$$

$$D^{(a)}(v)\Phi_W^{(a,a')}(v)=\Phi_S^{(a,a')}(0)D^{(a')}(v) \qquad \text{(A.10)}$$

we have

$$P(a_1,\ldots,a_m)=\frac{1}{Z}\,\mathrm{tr}_{\mathcal{H}_{l,a_1+\rho}}\,D^{(a_1)}(v)C^{(a_1)}(v)B^{(a_1)}(v)A^{(a_1)}(v)$$
$$\times\Phi_E^{(a_1,a_2)}(0)\cdots\Phi_E^{(a_{m-1},a_m)}(0)\Phi_S^{(a_m,a_{m-1})}(0)\cdots\Phi_S^{(a_2,a_1)}(0)\,. \qquad \text{(A.11)}$$

We wish to identify the space $\mathcal{H}_{l,k}$ and the operators $\Phi_*^{(a',a)}(u)$ $(*=E,S)$ acting on it with a certain boson Fock space and bosonized vertex operators. By a routine argument, we can derive the following commutation relations:

$$\Phi_E^{(a_1,a_2)}(v_1)\Phi_E^{(a_2,a_3)}(v_2)=\sum_a W\left(\begin{matrix}a_1 & a\\ a_2 & a_3\end{matrix}\middle| v_1-v_2\right)\Phi_E^{(a_1,a)}(v_2)\Phi_E^{(a,a_3)}(v_1)$$
$$\Phi_S^{(a_1,a_2)}(v_1)\Phi_E^{(a_2,a_3)}(v_2)=\sum_a W\left(\begin{matrix}a_2 & a_1\\ a_3 & a\end{matrix}\middle| v_1+v_2\right)\Phi_E^{(a_1,a)}(v_2)\Phi_S^{(a,a_3)}(v_1) \qquad \text{(A.12)}$$
$$\Phi_S^{(a_1,a_2)}(v_1)\Phi_S^{(a_2,a_3)}(v_2)=\sum_a W\left(\begin{matrix}a_3 & a_2\\ a & a_1\end{matrix}\middle| v_2-v_1\right)\Phi_S^{(a_1,a)}(v_2)\Phi_S^{(a,a_3)}(v_1)\,.$$

These relations are satisfied by the bosonized vertex operators $\phi_\mu(v)$ (3.12) and $\phi_\mu^{*(n-1)}(v)$ (3.18) if we set

$$\Phi_E^{(a_1,a_2)}(v)=\phi_\mu(v) \qquad (a_1\overset{\mu}{\leftarrow}a_2) \qquad \text{(A.13)}$$

$$\Phi_S^{(a_1,a_2)}(v)=(-1)^{\mu-1}\phi_\mu^{*(n-1)}\left(\frac{n}{2}-v\right)\prod_{\kappa<\lambda}[\pi_{\kappa\lambda}]^{-1} \qquad (a_1\overset{\mu}{\rightarrow}a_2)\,. \qquad \text{(A.14)}$$

However, this is not the correct identification because the spaces $\mathcal{H}_{l,k}$ and $\mathcal{F}_{l,k}$ have different characters. As discussed in section 4, we expect that the BRST cohomology of certain complex consisting of the spaces $\mathcal{F}_{l,k}$ provides the correct identification of the space $\mathcal{H}_{l,k}$. Under this assumption we can write down an integral formula for the local probabilities. We will not enter in detail.

Appendix B. Proof of the commutation relations

The operator $\phi_\mu(v)$ is given by (recall $z_j=x^{2v_j}$)

$$\phi_\mu(v)=\oint\prod_{j=1}^{\mu-1}\frac{\mathrm{d}z}{2\pi\mathrm{i}z_j}\eta_1(v_0)\xi_1(v_1)\cdots\xi_{\mu-1}(v_{\mu-1})\prod_{j=1}^{\mu-1}f(v_j-v_{j-1},\pi_{j,\mu})\,. \qquad \text{(B.1)}$$

For the above integral, we call $\eta_1(v_0)\xi_1(v_1)\cdots\xi_{\mu-1}(v_{\mu-1})$ the operator part, and $\prod_{j=1}^{\mu-1} f(v_j - v_{j-1}, \pi_{j,\mu})$ the coefficient part. We will use these names for similar integrals. Note that

$$\pi_\alpha \exp\left(-\mathrm{i}\sqrt{\frac{r-1}{r}}Q_\beta\right) = \exp\left(-\mathrm{i}\sqrt{\frac{r-1}{r}}Q_\beta\right)(\pi_\alpha + (1-r)\langle\alpha,\beta\rangle)\ . \tag{B.2}$$

Therefore, the operator parts and coefficient parts are not commutative. Unless otherwise stated, we keep coefficient parts to the right of operator parts.

We represent $\prod_{j=1}^{\mu-1} f(v_j - v_{j-1}, \pi_{j,\mu})$ by the diagram

$$v_{\mu-1} \xrightarrow{\pi_{\mu-1,\mu}} v_{\mu-2} \xrightarrow{\pi_{\mu-2,\mu}} \cdots \xrightarrow{\pi_{2,\mu}} v_1 \xrightarrow{\pi_{1,\mu}} v_0. \tag{B.3}$$

Using the commutation relations (C.1)–(C.7), we are to prove

$$\phi_\mu(v_1)\phi_\mu(v_2) = r_1(v_1 - v_2)\phi_\mu(v_2)\phi_\mu(v_1) \tag{B.4}$$

$$\begin{aligned}\phi_\mu(v_1)\phi_\nu(v_2) = r_1(v_1 - v_2)\big\{&\phi_\nu(v_2)\phi_\mu(v_1)b(v_1 - v_2, \pi_{\mu,\nu}) \\ &+\phi_\mu(v_2)\phi_\nu(v_1)c(v_1 - v_2, \pi_{\mu,\nu})\big\} \qquad (\mu \neq \nu)\end{aligned} \tag{B.5}$$

where

$$b(v,w) = \frac{[v][w-1]}{[v-1][w]} \qquad c(v,w) = \frac{[v-w][1]}{[v-1][w]}\,. \tag{B.6}$$

Consider an integral of the form

$$\oint \frac{\mathrm{d}z}{2\pi\mathrm{i}z_j}\frac{\mathrm{d}z'}{2\pi\mathrm{i}z'_j}\xi_j(v_j)\xi_j(v'_j)F(v_j, v'_j) \tag{B.7}$$

where the integration contours for z_j and z'_j are the same. Due to equation (C.7), this integral is equal to

$$\oint \frac{\mathrm{d}z}{2\pi\mathrm{i}z_j}\frac{\mathrm{d}z'}{2\pi\mathrm{i}z'_j}\xi_j(v_j)\xi_j(v'_j)h(v'_j - v_j)F(v'_j, v_j)\,. \tag{B.8}$$

Observing this we define 'weak equality' in the following sense. Suppose two functions $F(v_j, v'_j)$ and $G(v_j, v'_j)$ are coupled to $\xi_j(v_j)\xi_j(v'_j)$ in integrals. We say they are equal in weak sense if

$$G(v_j, v'_j) + h(v'_j - v_j)G(v'_j, v_j) = F(v_j, v'_j) + h(v'_j - v_j)F(v'_j, v_j)\,. \tag{B.9}$$

We write

$$G(v_j, v'_j) \sim F(v_j, v'_j) \tag{B.10}$$

showing weak equality. To prove the equalities (B.4) and (B.5), it is enough to prove the equalities of the coefficient parts in this weak sense.

First we will prove (B.4). By using (C.1-C.7) and (B.2), we can rearrange the operator part as $\eta_1(v_0)\eta_1(v'_0)\xi_1(v_1)\xi_1(v'_1)\cdots\xi_{\mu-1}(v_{\mu-1})\ i_{\mu-1}(v'_{\mu-1})$, and then get the coefficient part represented by

$$\begin{tikzcd} v_{\mu-1} \arrow[r, "\pi_{\mu-1,\mu}-1"] \arrow[dr, "0"] & v_{\mu-2} \arrow[r, "\pi_{\mu-2,\mu}-1"] \arrow[dr, "0"] & \cdots \arrow[r, "\pi_{2,\mu}-1"] \arrow[dr, "0"] & v_1 \arrow[r, "\pi_{1,\mu}-1"] \arrow[dr, "0"] & v_0 \\ v'_{\mu-1} \arrow[r, "\pi_{\mu-1,\mu}"] & v'_{\mu-2} \arrow[r, "\pi_{\mu-2,\mu}"] & \cdots \arrow[r, "\pi_{2,\mu}"] & v'_1 \arrow[r, "\pi_{1,\mu}"] & v'_0\,. \end{tikzcd} \tag{B.11}$$

We want to show that this is invariant in weak sense when v_0 and v_0' are exchanged. This follows immediately by induction from the weak equality

$$f(v_1 - v_0, w-1) f(v_1' - v_0', w) f(v_1 - v_0', 0) \tag{B.12}$$

$$\sim f(v_1 - v_0', w-1) f(v_1' - v_0, w) f(v_1 - v_0, 0)\,. \tag{B.13}$$

Next we prove (B.5) for $\mu < \nu$. The case $\mu > \nu$ is similar. The equality follows from the weak equality $(A) + (B) + (C) \sim 0$ where

$$(A) = \begin{tikzcd} v_\mu' \arrow[r, "\pi_{\mu,\nu}"] & v_{\mu-1}' \arrow[r, "\pi_{\mu-1,\nu}"] & \cdots \arrow[r, "\pi_{3,\nu}"] & v_2' \arrow[r, "\pi_{2,\nu}"] & v_1' \arrow[r, "\pi_{1,\nu}"] & v_0' \\ & v_{\mu-1} \arrow[r, "\pi_{\mu-1,\mu}"'] \arrow[ur, "0"'] & \cdots \arrow[r, "\pi_{3,\mu}"'] \arrow[ur, "0"'] & v_2 \arrow[r, "\pi_{2,\mu}"'] \arrow[ur, "0"'] & v_1 \arrow[r, "\pi_{1,\mu}"'] \arrow[ur, "0"'] & v_0 \end{tikzcd}$$

$$(B) = -b(v_0 - v_0', \pi_{\mu,\nu})$$

$$\times \begin{tikzcd} v_\mu' \arrow[r, "\pi_{\mu,\nu}+1"] \arrow[dr, "0"] & v_{\mu-1} \arrow[r, "\pi_{\mu-1,\nu}"] \arrow[dr, "0"] & \cdots \arrow[r, "\pi_{3,\nu}"] \arrow[dr, "0"] & v_2 \arrow[r, "\pi_{2,\nu}"] \arrow[dr, "0"] & v_1 \arrow[r, "\pi_{1,\nu}"] \arrow[dr, "0"] & v_0' \\ & v_{\mu-1}' \arrow[r, "\pi_{\mu-1,\mu}"'] & \cdots \arrow[r, "\pi_{3,\mu}"'] & v_2' \arrow[r, "\pi_{2,\mu}"'] & v_1' \arrow[r, "\pi_{1,\mu}"'] & v_0 \end{tikzcd}$$

$$(C) = -c(v_0 - v_0', \pi_{\mu,\nu})$$

$$\times \begin{tikzcd} v_\mu' \arrow[r, "\pi_{\mu,\nu}"] & v_{\mu-1}' \arrow[r, "\pi_{\mu-1,\nu}"] & \cdots \arrow[r, "\pi_{3,\nu}"] & v_2' \arrow[r, "\pi_{2,\nu}"] & v_1' \arrow[r, "\pi_{1,\nu}"] & v_0 \\ & v_{\mu-1} \arrow[r, "\pi_{\mu-1,\mu}"'] \arrow[ur, "0"'] & \cdots \arrow[r, "\pi_{3,\mu}"'] \arrow[ur, "0"'] & v_2 \arrow[r, "\pi_{2,\mu}"'] \arrow[ur, "0"'] & v_1 \arrow[r, "\pi_{1,\mu}"'] \arrow[ur, "0"'] & v_0'\,. \end{tikzcd}$$

We prove this by induction starting from the equality

$$f(v_1' - v_0', w) = b(v_0 - v_0', w) f(v_1' - v_0', w+1) f(v_1' - v_0, 0)$$

$$+ c(v_0 - v_0', w) f(v_1' - v_0, w)\,.$$

Using the induction hypothesis we modify (B) to $(A') + (C')$ where

$$(A') = -\frac{b(v_0 - v_0', \pi_{\mu,\nu})}{b(v_1' - v_1, \pi_{\mu,\nu})}$$

$$\times \begin{tikzcd} v_\mu' \arrow[r, "\pi_{\mu,\nu}"] & v_{\mu-1}' \arrow[r, "\pi_{\mu-1,\nu}"] & \cdots \arrow[r, "\pi_{3,\nu}"] & v_2' \arrow[r, "\pi_{2,\nu}"] & v_1 \arrow[r, "\pi_{1,\nu}"] \arrow[dr, "0"] & v_0' \\ & v_{\mu-1} \arrow[r, "\pi_{\mu-1,\mu}"'] \arrow[ur, "0"'] & \cdots \arrow[r, "\pi_{3,\mu}"'] \arrow[ur, "0"'] & v_2 \arrow[r, "\pi_{2,\mu}"'] \arrow[ur, "0"'] & v_1' \arrow[r, "\pi_{1,\mu}"'] & v_0 \end{tikzcd}$$

$$(C') = \frac{b(v_0 - v_0', \pi_{\mu,\nu}) c(v_1' - v_1, \pi_{\mu,\nu})}{b(v_1' - v_1, \pi_{\mu,\nu})}$$

$$\times \begin{tikzcd} v_\mu' \arrow[r, "\pi_{\mu,\nu}"] & v_{\mu-1}' \arrow[r, "\pi_{\mu-1,\nu}"] & \cdots \arrow[r, "\pi_{3,\nu}"] & v_2' \arrow[r, "\pi_{2,\nu}"] & v_1' \arrow[r, "\pi_{1,\mu}"] & v_0 \\ & v_{\mu-1} \arrow[r, "\pi_{\mu-1,\mu}"'] \arrow[ur, "0"'] & \cdots \arrow[r, "\pi_{3,\mu}"'] \arrow[ur, "0"'] & v_2 \arrow[r, "\pi_{2,\mu}"'] \arrow[ur, "0"'] & v_1 \arrow[r, "\pi_{1,\nu}"'] \arrow[ur, "0"'] & v_0'\,. \end{tikzcd}$$

Noting that $b(v_1' - v_1, w)h(v_1' - v_1) = b(v_1 - v_1', w)$ we can exchange v_1 and v_1' in (A'). Let (A'') be the term we thus obtain. Note that $(A') \sim (A'')$. Using the equality

$$\begin{aligned} &b(v_1' - v_1, w) f(v_1 - v_0', 0) - b(v_0 - v_0', w) f(v_1' - v_0, 0) \\ &\qquad = \frac{[w-1][1][v_0 - v_0' + v_1 - v_1'][v_0 - v_1 - \frac{1}{2}][v_0' - v_1' + \frac{1}{2}]}{[w][v_1' - v_1 - 1][v_0 - v_0' - 1][v_1' - v_0 - \frac{1}{2}][v_1 - v_0' - \frac{1}{2}]} \end{aligned} \tag{B.14}$$

we have

$$(A) + (A'') = -\frac{[1][v_0 - v_0' + v_1 - v_1'][v_1 - v_0 + \frac{1}{2}][v_1' - v_0' - \frac{1}{2}]}{[v_1 - v_1'][v_0 - v_0' - 1][v_1' - v_0 - \frac{1}{2}][v_1 - v_0' - \frac{1}{2}]}$$

$$\times \begin{tikzcd} v_\mu' \arrow[r, "\pi_{\mu,\nu}"] & v_{\mu-1}' \arrow[r, "\pi_{\mu-1,\nu}"] & \cdots \arrow[r, "\pi_{3,\nu}"] & v_2' \arrow[r, "\pi_{2,\nu}"] & v_1' \arrow[r, "\pi_{1,\nu}"] & v_0' \\ & v_{\mu-1} \arrow[r, "\pi_{\mu-1,\mu}"'] \arrow[ur, "0"'] & \cdots \arrow[r, "\pi_{3,\mu}"'] \arrow[ur, "0"'] & v_2 \arrow[r, "\pi_{2,\mu}"'] \arrow[ur, "0"'] & v_1 \arrow[r, "\pi_{1,\mu}"'] & v_0\,. \end{tikzcd}$$

Using the equality

$$\begin{aligned} &\frac{c(v_1' - v_1, w_1)}{b(v_1' - v_1, w_1)} f(v_1' - v_0, w_2) f(v_1 - v_0', w_1 + w_2) \\ &\qquad - \frac{c(v_0 - v_0', w_1)}{b(v_0 - v_0', w_1)} f(v_1 - v_0', w_2) f(v_1' - v_0, w_1 + w_2) \\ &\qquad = \frac{[w_1][1][v_0 - v_0' + v_1 - v_1'][v_1 - v_0 + \frac{1}{2} - w_2][v_1' - v_0' + \frac{1}{2} - w_1 - w_2]}{[w_1 - 1][v_0 - v_0'][v_1 - v_1'][v_1' - v_0 - \frac{1}{2}][v_1 - v_0' - \frac{1}{2}]} \end{aligned}$$

we have

$$(C) + (C') = \frac{[1][v_0 - v_0' + v_1 - v_1'][v_1 - v_0 + \frac{1}{2} - \pi_{1,\mu}][v_1' - v_0' + \frac{1}{2} - \pi_{1,\nu}]}{[v_0 - v_0' - 1][v_1 - v_1'][v_1' - v_0 - \frac{1}{2}][v_1 - v_0' - \frac{1}{2}]}$$

$$\times \begin{tikzcd} v_\mu' \arrow[r, "\pi_{\mu,\nu}"] & v_{\mu-1}' \arrow[r, "\pi_{\mu-1,\nu}"] & \cdots \arrow[r, "\pi_{3,\nu}"] & v_2' \arrow[r, "\pi_{2,\nu}"] & v_1' & \\ & v_{\mu-1} \arrow[r, "\pi_{\mu-1,\mu}"'] \arrow[ur, "0"'] & \cdots \arrow[r, "\pi_{3,\mu}"'] \arrow[ur, "0"'] & v_2 \arrow[r, "\pi_{2,\mu}"'] \arrow[ur, "0"'] & v_1 \arrow[r, "0"'] & v_0\,. \end{tikzcd}$$

Comparing these expressions, we have $(A) + (A'') + (C) + (C') = 0$.

Appendix C. Proof of theorems 3.2 and 3.3

In this appendix we give a proof of the bosonization formulae for the operators $\phi_\mu^{*(m)}(v)$. For convenience we list below formulae for the contractions of $\eta_j(v) = U_{\omega_j}(z)$, $\xi_j(v) = U_{-\alpha_j}(z)$:

$$U_{\omega_1}(z_1) U_{\omega_m}(z_2) = z_1^{((r-1)/r)(n-m)/n}\ g_m(z_2/z_1) : U_{\omega_1}(z_1) U_{\omega_m}(z_2) : \tag{C.1}$$

$$U_{\omega_m}(z_2) U_{\omega_1}(z_1) = z_2^{((r-1)/r)(n-m)/n}\ g_m(z_1/z_2) : U_{\omega_1}(z_1) U_{\omega_m}(z_2) : \tag{C.2}$$

$$U_{-\alpha_j}(z_1) U_{\omega_j}(z_2) = z_1^{-(r-1)/r}\ s(z_2/z_1) : U_{-\alpha_j}(z_1) U_{\omega_j}(z_2) : \tag{C.3}$$

$$U_{\omega_j}(z_2) U_{-\alpha_j}(z_1) = z_2^{-(r-1)/r}\ s(z_1/z_2) : U_{-\alpha_j}(z_1) U_{\omega_j}(z_2) : \tag{C.4}$$

$$U_{-\alpha_j}(z_1)U_{-\alpha_{j+1}}(z_2) = z_1^{-(r-1)/r}\, s(z_2/z_1) : U_{-\alpha_j}(z_1)U_{-\alpha_{j+1}}(z_2) : \tag{C.5}$$

$$U_{-\alpha_{j+1}}(z_2)U_{-\alpha_j}(z_1) = z_2^{-(r-1)/r}\, s(z_1/z_2) : U_{-\alpha_j}(z_1)U_{-\alpha_{j+1}}(z_2) : \tag{C.6}$$

$$U_{-\alpha_j}(z_1)U_{-\alpha_j}(z_2) = z_1^{2(r-1)/r}\, t(z_2/z_1) : U_{-\alpha_j}(z_1)U_{-\alpha_j}(z_2) : \,. \tag{C.7}$$

Here

$$s(z) = \frac{(x^{2r-1}z; x^{2r})_\infty}{(xz; x^{2r})_\infty} \tag{C.8}$$

$$t(z) = (1-z)\frac{(x^2z; x^{2r})_\infty}{(x^{2r-2}z; x^{2r})_\infty}\,. \tag{C.9}$$

For all other combinations except $U_{\omega_j}(z_1)U_{\omega_k}(z_2)$, we have $XY =: XY :$.

Let us assign a weight to these operators by setting wt $U_{\omega_j}(z) = \omega_j$, wt $U_{-\alpha_j}(z) = -\alpha_j$ and $\mathrm{wt}(XY) = \mathrm{wt}(X) + \mathrm{wt}(Y)$. Then $\mathrm{wt}\,\phi_\mu(v) = \bar{\epsilon}_\mu$, $\mathrm{wt}\,\bar{\phi}^{*(m-1)}_\mu(v) = \omega_m - \bar{\epsilon}_\mu$. It is useful to note that

$$\pi_{\mu\nu} X = X\left(\pi_{\mu\nu} + (1-r)\langle\epsilon_\mu - \epsilon_\nu, \mathrm{wt}\, X\rangle\right)\,. \tag{C.10}$$

Lemma C.1. For $1 \leqslant \mu \leqslant m$ we have

$$\phi_\mu(v)\bar{\phi}^{*(m-1)}_\mu(v') = -r_{m-1}(v-v') \times \sum_{\nu=1}^{m} \bar{\phi}^{*(m-1)}_\nu(v')\phi_\nu(v)\frac{[v-v'-\frac{m}{2}+1-\pi_{\mu\nu}]}{[v-v'-\frac{m}{2}]}\prod_{\substack{1\leqslant\kappa\leqslant m\\ \kappa\neq\nu}}\frac{[1-\pi_{\mu\kappa}]}{[\pi_{\nu\kappa}]}. \tag{C.11}$$

Proof. Set $v_0 = v$, $v_m = v'$. As before we write $z_j = x^{2v_j}$.

Using equations (3.12), (3.15) and (C.10), we find

$$c_m\phi_\mu(v)\bar{\phi}^{*(m-1)}_\mu(v') = (-1)^{m-1}\prod_{j=1}^{m-1}\oint\frac{\mathrm{d}z_j}{2\pi\mathrm{i}z_j}\eta_1(v_0)\xi_1(v_1)\cdots\xi_{m-1}(v_{m-1})\eta_{m-1}(v_m) \times\prod_{j=1}^{m} f(v_j - v_{j-1}, 1-\pi_{\mu j})\,. \tag{C.12}$$

Here we used $[u+r] = -[u]$. Similarly we have

$$c_m r_{m-1}(v-v')\bar{\phi}^{*(m-1)}_\nu(v')\phi_\nu(v) = -\prod_{j=1}^{m-1}\oint\frac{\mathrm{d}z_j}{2\pi\mathrm{i}z_j}\eta(v_0)\xi_1(v_1)\cdots\xi_{m-1}(v_{m-1})\eta_{m-1}(v_m)\prod_{j=1}^{m} f(v_j - v_{j-1}, \pi_{j\nu})\,. \tag{C.13}$$

The lemma is proved if we show that (C.11) is valid at the level of the integrands of (C.12), (C.13). This amounts to showing that

$$(-1)^{m-1}\prod_{j=1}^{m} f(v_j - v_{j-1}, 1-\pi_{\mu j}) = \sum_{\nu=1}^{m}\prod_{j=1}^{m} f(v_j - v_{j-1}, \pi_{j\nu})\frac{[v_0 - v_m - \frac{m}{2}+1-\pi_{\mu\nu}]}{[v_0 - v_m - \frac{m}{2}]}\prod_{\substack{1\leqslant\kappa\leqslant m\\ \kappa\neq\nu}}\frac{[1-\pi_{\mu\kappa}]}{[\pi_{\nu\kappa}]}\,. \tag{C.14}$$

Consider the function

$$F(u)=\left(\prod_{j=1}^{m}\frac{[v_j-v_{j-1}+\frac{1}{2}-\pi_j+u]}{[-\pi_j+u]}\right)\frac{[v_0-v_m-\frac{m}{2}+1-\pi_\mu+u]}{[1-\pi_\mu+u]}\,.$$

This is an elliptic function in u. Setting the sum of its residues to zero, we obtain

$$0=(-1)^m\prod_{j=1}^{m}\left(\frac{[v_j-v_{j-1}-\frac{1}{2}+\pi_{\mu j}]}{[1-\pi_{\mu j}]}\right)\left[v_0-v_m-\frac{m}{2}\right]$$
$$+\sum_{\nu=1}^{m}\left(\prod_{j=1}^{m}[v_j-v_{j-1}+\frac{1}{2}+\pi_{\nu j}]\right)\frac{[v_0-v_m-\frac{m}{2}+1-\pi_{\mu\nu}]}{[1-\pi_{\mu\nu}]}\prod_{\substack{1\leqslant\kappa\leqslant m\\ \kappa\neq\nu}}[\pi_{\nu\kappa}]^{-1}$$

which is the desired identity (C.14). □

Lemma C.2.

$$\eta_1\left(v-\frac{m-1}{2}\right)\xi_1\left(v-\frac{m-2}{2}\right)\cdots\xi_{m-1}(v)\eta_{m-1}\left(v+\frac{1}{2}\right)=(x^{2r};x^{2r})_\infty^{3(m-1)}c_m\eta_m(v)\,. \tag{C.15}$$

This can be verified by a direct calculation using equations (C.1)–(C.7).

Lemma C.3.

$$\sum_{\mu=1}^{m}\phi_\mu\left(v-\frac{m-1}{2}\right)\bar{\phi}_\mu^{*(m-1)}\left(v+\frac{1}{2}\right)\prod_{\substack{1\leqslant\lambda\leqslant m\\ \lambda\neq\mu}}[\pi_{\mu\lambda}]^{-1}=\eta_m(v)=c_{m+1}\bar{\phi}_{m+1}^{*(m)}(v)\,. \tag{C.16}$$

Proof. Using equation (C.12) we find that the left-hand side becomes

$$c_m^{-1}\sum_{\mu=1}^{m}(-1)^{m-1}\oint_{C_\mu}\prod_{j=1}^{m-1}\frac{\mathrm{d}z_j}{2\pi\mathrm{i}z_j}\eta_1(v_0)\xi_1(v_1)\cdots\xi_{m-1}(v_{m-1})\eta_{m-1}(v_m)F_\mu(v_1,\ldots,v_{m-1}) \tag{C.17}$$

where

$$F_\mu(v_1,\ldots,v_{m-1})=\prod_{1\leqslant j\,(\neq\mu)\leqslant m}\frac{f(v_j-v_{j-1},1-\pi_{\mu j})}{[\pi_{\mu j}]}$$

and

$$v_0=v-\frac{m-1}{2}\qquad v_m=v+\frac{1}{2}\,.$$

Note that $f(v,1)=1$. The contour C_μ is chosen as

$$\begin{aligned} C_\mu:|z_j|&=x^{-m+j+1}(|z|+j\varepsilon) &&(1\leqslant j\leqslant\mu-1)\\ |z_j|&=x^{-m+j+1}(|z|-(m-j)\varepsilon) &&(\mu\leqslant j\leqslant m-1)\end{aligned} \tag{C.18}$$

where $\varepsilon>0$ is a small number.

Consider now the elliptic function

$$F(u)=\prod_{j=1}^{m}\frac{[v_j-v_{j-1}-\frac{1}{2}+u-\pi_j]}{[v_j-v_{j-1}-\frac{1}{2}][u-\pi_j]}\,.$$

Applying the residue theorem to $F(u)$, we find

$$\sum_{\mu=1}^{m} F_\mu(v_1, \dots, v_{m-1}) = 0\,. \tag{C.19}$$

In the neighbourhood of the contour C_μ, the poles of the integrand of (C.17) are those of F_μ. In particular, the only pole for z_1 is $x^{-m+2}z$. Let us change the contour for z_1 into $|z_1| = x^{-m+2}(|z| - \varepsilon)$ for $\mu \geqslant 2$. Then the resulting integrals can be taken on a contour common to all μ. Due to equation (C.19), the sum gives zero. Therefore the right-hand side of (C.17) can be replaced by its residue at $z_1 = x^{-m+2}z$:

$$\begin{aligned}
&c_m^{-1}\sum_{\mu=2}^{m}(-1)^{m-1}\oint_{C'_\mu}\prod_{j=2}^{m-1}\frac{dz_j}{2\pi i z_j}\eta_1(v_0)\xi_1(v_1)\cdots\xi_{m-1}(v_{m-1})\eta_{m-1}(v_m)\\
&\qquad\times \mathrm{Res}_{v_1=v-(m-2)/2}\, F_\mu(v_1,\dots,v_{m-1})\frac{dz_1}{z_1}\\
&\quad= c_m^{-1}A\sum_{\mu=2}^{m}(-1)^{m-2}\oint_{C'_\mu}\prod_{j=2}^{m-1}\eta_1(v_0)\xi_1(v_1)\cdots\xi_{m-1}(v_{m-1})\eta_{m-1}(v_m)\\
&\qquad\times F'_\mu(v_2,\dots,v_{m-1})
\end{aligned}$$

where now $v_0 = v-(m-1)/2$, $v_1 = v-(m-2)/2$ and

$$F'_\mu(v_2,\dots,v_{m-1}) = \prod_{2\leqslant j(\neq\mu)\leqslant m}\frac{f(v_j - v_{j-1}, 1-\pi_{\mu j})}{[\pi_{\mu j}]}$$

$$A = -\,\mathrm{Res}_{v=0}\,\frac{1}{[v]}\frac{dz}{z} = \frac{1}{(x^{2r};x^{2r})_\infty^3}\,.$$

The contour C'_μ is given by (C.18) with $j \geqslant 2$. The function F'_μ and the contour C'_μ have the same structure as F_μ and C_μ, except that the number of integration variables is one less. Repeating this process $m-1$ times, we arrive at the result

$$c_m^{-1}A^{m-1}\eta_1\left(v-\frac{m-1}{2}\right)\xi_1\left(v-\frac{m-2}{2}\right)\cdots\xi_{m-1}(v)\eta_{m-1}\left(v+\frac{1}{2}\right)\,.$$

Equation (C.16) now follows from lemma C.2. □

Lemma C.4. If $\mu \leqslant \nu$, then

$$\phi_\mu(v)\bar{\phi}_\nu^{*(n-1)}\left(v-\frac{n}{2}\right) = A_\mu\delta_{\mu\nu}\times \mathrm{id}\,. \tag{C.20}$$

where

$$A_\mu = (-1)^{n-1}\frac{1}{\Theta_{x^{2r}}(x^2)}\prod_{k=1}^{n}[1+\pi_{k\mu}]\,.$$

Proof. Suppose that $\mu < \nu$. Then it is easy to see that

$$\phi_\mu(v)\bar{\phi}_\nu^{*(n-1)}(v') = r_{n-1}(v-v')\bar{\phi}_\nu^{*(n-1)}(v')\phi_\mu(v)\,.$$

As $v' \to v - n/2$, $\bar{\phi}_\nu^{*(n-1)}(v')\phi_\mu(v)$ is regular, while $r_{n-1}(v-v')$ has a simple zero. This shows (C.20) for $\mu < \nu$.

Suppose next that $\mu=\nu$. Then

$$\phi_\mu(v)\bar\phi^{*(n-1)}_\mu(v')=-c_n^{-1}r_{n-1}(v-v')$$
$$\times\prod_{j=1}^{n-1}\oint\frac{\mathrm{d}z_j}{2\pi\mathrm{i}z_j}\eta_{n-1}(v')\xi_{n-1}(v_{n-1})\cdots\xi_1(v_1)\eta_1(v)\prod_{j=1}^{n}f(v_{j-1}-v_j,\pi_{\mu j})\,.$$

The contour is

$$|z_j|=x^{-j}(|z|-j\varepsilon)\qquad(1\leqslant j\leqslant\mu-1)$$
$$|z_j|=x^{n-j}(|z'|+(n-j)\varepsilon)\qquad(\mu\leqslant j\leqslant n-1)$$
$$|z_\mu|<x^{-1}|z_{\mu-1}|\,.$$

As $v'\to v-n/2$, the contour is pinched but $r_{n-1}(v-v')$ has a zero. The limit is then calculated by successively taking the residues at $v_j=v_{j-1}-1/2$ for $1\leqslant j\leqslant\mu-1$ and $v_j=v_{j+1}+1/2$ for $\mu\leqslant j\leqslant n-1$. Calculating similarly as in lemma C.3 we find

$$(-1)^n c_n^{-1}(x^{2r};x^{2r})_\infty^{-3(n-1)}\lim_{v\to0}\frac{r_{n-1}(v+\frac{n}{2})}{[v]}$$
$$\times\eta_{n-1}\left(v-\frac{n}{2}\right)\xi_{n-1}\left(v-\frac{n-1}{2}\right)\cdots\xi_1\left(v-\frac{1}{2}\right)\eta_1(v)\prod_{j=1}^{n}[1+\pi_{j\mu}]$$
$$=(-1)^{n-1}\frac{1}{\Theta_{x^{2r}}(x^2)}\eta_n\left(v-\frac{1}{2}\right)\prod_{j=1}^{n}[1+\pi_{j\mu}]\,.$$

Noting that $\eta_n(v)=U_{\omega_n}(z)=1$, we obtain the lemma. □

Proof of theorem 3.2. Set

$$\phi^{*(m)}_\mu(v)=\tilde\phi^{*(m)}_\mu(v)\prod_{\substack{1\leqslant\kappa<\lambda\leqslant m+1\\ \kappa,\lambda\neq\mu}}[\pi_{\kappa\lambda}]\,.$$

We show that

$$\tilde\phi^{*(m)}_\mu(v)=\bar\phi^{*(m)}_\mu(v)\tag{C.21}$$

by induction on m. The case $m=1$ is trivially true. Consider first the case $\mu=m+1$. By the definition, we have

$$\tilde\phi^{*(m)}_{m+1}(v)=c_{m+1}^{-1}\sum_{\mu=1}^{m}\phi_\mu\left(v-\frac{m-1}{2}\right)\tilde\phi^{*(m-1)}_\mu\left(v+\frac{1}{2}\right)\prod_{\substack{1\leqslant\lambda\leqslant m\\ \lambda\neq\mu}}[\pi_{\mu\lambda}]^{-1}\,.$$

Using the induction hypothesis and lemma C.3, we conclude that $\tilde\phi^{*(m)}_{m+1}(v)=\bar\phi^{*(m)}_{m+1}(v)$.

Recall that $\tilde\phi^{*(m)}_\mu(v)$ satisfy the same commutation relation (C.11) as $\bar\phi^{*(m)}_\mu(v)$. Taking $\mu=m+1$ and computing $\phi_{m+1}(v)(\tilde\phi^{*(m)}_{m+1}(v)-\bar\phi^{*(m)}_{m+1}(v))$ we find

$$0=\sum_{\nu=1}^{m}(\tilde\phi^{*(m)}_\nu(v')-\bar\phi^{*(m)}_\nu(v'))\phi_\nu(v)\frac{[v-v'-\frac{m+1}{2}+1-\pi_{\mu\nu}]}{[v-v'-\frac{m+1}{2}]}\prod_{\substack{1\leqslant\kappa\leqslant m+1\\ \kappa\neq\nu}}\frac{[1-\pi_{\mu\kappa}]}{[\pi_{\nu\kappa}]}\,.\tag{C.22}$$

Multiplying $\phi^{*(n-1)}_m(v-n/2)$ from the right, using lemma C.4, we have

$$0=\tilde\phi^{*(m)}_m(v')-\bar\phi^{*(m)}_m(v')\,.$$

Substituting back to (C.22) and multiplying $\phi^{*(n-1)}_{m-1}(v - n/2)$ from the right we obtain

$$0 = \tilde{\phi}^{*(m)}_{m-1}(v') - \bar{\phi}^{*(m)}_{m-1}(v') \,.$$

Continuing this process we get (C.21). □

Proof of theorem 3.3. It suffices to prove an equivalent statement (3.20). This follows as a special case of (C.16) with $m = n$. □

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